Antibrackets, Supersymmetric σ-Model and Localization

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We consider supersymmetrization of Hamiltonian dynamics via antibrackets for systems whose Hamiltonian generates an isometry of the phase space. We find that the models are closely related to the supersymmetric non-linear σ-model. We interpret the corresponding path integrals in terms of super loop space equivariant cohomology. It turns out that they can be evaluated exactly using localizations techniques.

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Localization theorems (for a review see eg. [1]) state that certain infinite or finite dimensional integrals can be reduced to finite dimensional integrals or discrete sums, provided certain geometrical conditions are satisfied. This happens for example in topological theories where localization is achieved using Mathai-Quillen formalism [2]. On the other hand, the Duistermaat-Heckman theorem [3] can be applied to some Hamiltonian systems if the Hamiltonian generates isometries on the phase space. In this letter we shall consider supersymmetrized Hamiltonian mechanics using antibrackets and path integral localization techniques. The antibracket was introduced by Batalin and Fradkin [4] for the Lagrangian quantization of field theories with constraints. In this letter we use it to define odd symplectic structures on supermanifolds, which allows us to consider Hamiltonian systems. However, since the corresponding Hamiltonian is odd, it is necessary to construct an even symplectic structure and an even Hamiltonian to quantize the systems with path integrals. This turns out to be possible for Hamiltonians which generate isometries [5]. This condition enables the application of localization methods [6] based on the equivariant cohomology [7] to evaluate the corresponding path integrals exactly. These results can be interpreted as infinite dimensional generalizations of the Duistermaat-Heckman theorem [3].

This letter is organized as follows. First we consider formulation of Hamiltonian dynamics on a supermanifold using both odd and even symplectic structures. With the even structure we are able to consider the corresponding quantum mechanical partition functions. It turns out that the models are closely related to the supersymmetric non-linear σ-model. We interpret the pertinent path integrals using equivariant cohomology and evaluate them exactly by localization. The results turn out to be equivariant generalizations of familiar topological invariants.

We start by considering a Hamiltonian system (M, ω, H) where M is a phase space with a symplectic 2-form $\omega = \frac{1}{2}\omega_{ab}\mathrm{d}x^a \wedge \mathrm{d}x^b$ (in local coordinates) and H is the Hamiltonian. The symplectic structure determines the Poisson bracket

$$\{A, B\} = \omega(\chi_A, \chi_B) = \partial_a A \omega^{ab} \partial_b B . \tag{1}$$

Hamilton's equations of motion are given by $\dot{x}^a = \{x^a, H\} = \chi_H^a$. Here χ_H denotes the Hamiltonian vector field corresponding to H determined by the equation

$$\mathrm{d}H + \iota_H \omega = 0 \tag{2}$$

where ι_H denotes the contraction along χ_H.

We can also consider Hamiltonian dynamics on the supermanifold $\mathcal{M}$ associated to the cotangent bundle T^*M. The coordinates on $\mathcal{M}$ are denoted by $z^A = (x^a, \theta^a)$. We

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define an odd symplectic structure on $\mathcal{M}$ by introducing a non-degenerate odd symplectic 2-form

$$\Omega^1 = \mathrm{d}z^A \Omega^1_{AB} \mathrm{d}z^B \tag{3}$$

This 2-form determines the antibracket (odd Poisson bracket) whose grading and anti-symmetry properties are opposite to those of the ordinary graded Poisson bracket:

$$
\begin{aligned}
\epsilon(\mathcal{A}, \mathcal{B}) &= \epsilon(\mathcal{A}) + \epsilon(\mathcal{B}) + 1 \;, \\
(\mathcal{A}, \mathcal{B}) &= -(-1)^{(\epsilon_A+1)(\epsilon_B+1)}(\mathcal{B}, \mathcal{A}) \;, \\
(\mathcal{A}, \mathcal{BC}) &= (\mathcal{A}, \mathcal{B})\mathcal{C} + (-1)^{\epsilon_B(\epsilon_A+1)}\mathcal{B}(\mathcal{A}, \mathcal{C}) \\
0 &= (-1)^{(\epsilon_A+1)(\epsilon_C+1)}(\mathcal{A}, (\mathcal{B}, \mathcal{C})) + \text{cyclic perm} \;.
\end{aligned}
\tag{4}
$$

The 2-form Ω^1 can be written in local coordinates $z^A = (x^a, \theta^a)$ as

$$\Omega^1 = \omega_{ab}\mathrm{d}x^a \wedge \mathrm{d}\theta^b + \frac{\partial \omega_{ab}}{\partial x^c}\theta^c \mathrm{d}\theta^a \wedge \mathrm{d}\theta^b \tag{5}$$

and the antibracket becomes

$$(\mathcal{A}, \mathcal{B}) = \omega^{ab}\left(\frac{\partial^r \mathcal{A}}{\partial x^a}\frac{\partial^l \mathcal{B}}{\partial \theta^b} - \frac{\partial^r \mathcal{B}}{\partial x^a}\frac{\partial^l \mathcal{A}}{\partial \theta^b}\right) + \frac{\partial \omega^{ab}}{\partial x^c}\theta^c\frac{\partial^r \mathcal{A}}{\partial \theta^a}\frac{\partial^l \mathcal{B}}{\partial \theta^b} \tag{6}$$

where where $\mathcal{A}(x, \theta)$ and $\mathcal{B}(x, \theta)$ etc. are superfunctions on $\mathcal{M}$. The superscripts r and l denote right and left derivatives, respectively. In particular, we have the basic antibrackets

$$
\begin{aligned}
(x^a, x^b) &= 0 \;, \\
(x^a, \theta^b) &= \omega^{ab} \;, \\
(\theta^a, \theta^b) &= \frac{\partial \omega^{ab}}{\partial x^c}\theta^c \;.
\end{aligned}
\tag{7}
$$

We define a dynamical system on $\mathcal{M}$ by mapping the original Hamiltonian to an odd Hamiltonian with the function $\mathcal{F} = \frac{1}{2}\omega_{ab}\theta^a\theta^b$

$$\mathcal{Q}_H = (H, \mathcal{F}) = \frac{\partial H}{\partial x^a}\theta^a \;. \tag{8}$$

The antibracket coincides with the original bracket

$$(\mathcal{A}, \mathcal{Q}_H) = \{\mathcal{A}, H\} \;. \tag{9}$$

The corresponding equations of motion are

$$
\begin{aligned}
\dot{x}^a &= (x^a, \mathcal{Q}_H) = \chi^a_H \;, \\
\dot{\theta}^a &= (\theta^a, \mathcal{Q}_H) = \partial_b \chi^a_H \theta^b \;.
\end{aligned}
\tag{10}
$$

Using the antibracket we have thus found a supersymmetric generalization of the ordinary Hamiltonian dynamics.

From now on we assume that the original symplectic manifold M admits a Riemannian metric g for which χ_H is a Killing vector:

$$\mathcal{L}_H g = 0 \tag{11}$$

or in a component form

$$\chi_H^c \partial_c g_{ab} + \partial_a \chi_H^c g_{bc} + \partial_b \chi_H^c g_{ac} = 0 \ . \tag{12}$$

Then the function $\tilde{\mathcal{Q}}_H = \frac{1}{2} g_{ab} \chi_H^a \theta^b$ is an integral of motion for the antibracket

$$(\mathcal{Q}_H, \tilde{\mathcal{Q}}_H) = 0 \ . \tag{13}$$

$\tilde{\mathcal{Q}}_H$ also yields a bi-Hamiltonian structure on M with the second symplectic structure $\tilde{\omega}$ and Hamiltonian $H_2 = K$

$$
\begin{aligned}
(\mathcal{F}, \tilde{\mathcal{Q}}_H) &= \frac{1}{2} \tilde{\omega}_{ab} \theta^a \theta^b = \frac{1}{2} \left[\partial_a (g_{bc} \chi_H^c) - \partial_b (g_{bc} \chi_H^c) \right] \theta^a \theta^b \ , \\
(H, \tilde{\mathcal{Q}}_H) &= H_2 = \frac{1}{2} g_{ab} \chi_H^a \chi_H^b
\end{aligned}
\tag{14}
$$

Assuming that $\tilde{\omega}$ is non-degenerate we see that the equations of motions coincide

$$\frac{\partial K}{\partial x^a} = \tilde{\omega}_{ab} \omega^{bc} \frac{\partial H}{\partial x^c} \tag{15}$$

This means that the system is classically integrable.

We proceed to reformulate the dynamics on $\mathcal{M}$ using an even Poisson bracket. We want to do this since we need an even Hamiltonian and symplectic structure for path integral quantization. An even symplectic structure on $\mathcal{M}$ is given by the following supersymplectic 2-form [5]

$$\Omega_\alpha = \frac{1}{2} \left(\omega_{(\alpha)ab} + R_{abcd} \theta^c \theta^d \right) \mathrm{d}x^a \wedge \mathrm{d}x^b + \frac{1}{2} g_{ab} D\theta^a \wedge D\theta^b \ . \tag{16}$$

Here R is the Riemannian curvature of M and $D\theta^a = \mathrm{d}\theta^a + \Gamma_{bc}^a \theta^b \mathrm{d}x^c$ the covariant derivative on M with the metric connection

$$\Gamma_{bc}^a = \frac{1}{2} g^{ae} \left(\partial_b g_{ce} + \partial_c g_{be} - \partial_e g_{bc} \right) \ . \tag{17}$$

The subscript $\alpha = 0, 2$ refers to the symplectic structures $(H_0 = H, \omega_0 = \omega, M)$ and $(H_2, \omega_2 = \tilde{\omega}, M)$. The corresponding symplectic 1-forms are

$$\Theta_0 = \vartheta_a \mathrm{d}x^a + g_{ab} \theta^a D\theta^b$$

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$$\Theta_2 = g_{ab}\chi_H^b \mathrm{d}x^a + g_{ab}\theta^a D\theta^b \tag{18}$$

The 2-form Ω_α determines the even Poisson bracket on $\mathcal{M}$

$$[\mathcal{A},\mathcal{B}]_\alpha = \nabla_a\mathcal{A}\|\omega_{ab} + R_{abcd}\theta^c\theta^d\|^{-1}\nabla_b\mathcal{B} + g^{ab}\frac{\partial^r\mathcal{A}}{\partial\theta^a}\frac{\partial^l\mathcal{B}}{\partial\theta^b};, \tag{19}$$

where

$$\nabla_a = \partial_a - \Gamma_{ac}^b\theta^c\frac{\partial}{\partial\theta^b}. \tag{20}$$

The equations of motion for odd and even Poisson brackets on $\mathcal{M}$ coincide:

$$[z^A,\mathcal{H}_\alpha]_\alpha = (z^A,\mathcal{Q}_H) \tag{21}$$

where $\mathcal{H}_\alpha = H_\alpha + \tilde{\omega}$. The odd and even Poisson brackets provide a bi-Hamiltonian structure also on $\mathcal{M}$ and therefore the system is integrable also on $\mathcal{M}$.

Having found an even symplectic structure on $\mathcal{M}$ we proceed to quantize these systems by path integrals. We are interested in evaluating the partition functions of the form

$$Z = \mathrm{Str}\ \exp[-\mathrm{i}TH] = \int \mathcal{D}\mu\exp[\mathrm{i}S] \tag{22}$$

where H and S are the pertinent Hamiltonians and classical actions, respectively and $\mathcal{D}\mu$ denotes integration over the (super) loop space LM which consists of all T-periodic loops for both commuting and anticommuting variables. The integration measure is the Liouville measure determined by the symplectic structure (16). The system (M,H,ω) has the classical action

$$S = \int_0^T \mathrm{d}t[\vartheta_a\dot{x}^a - H] \tag{23}$$

and the corresponding partition function becomes

$$Z = \int \mathcal{D}x\mathrm{Pf}\ \|\omega_{ab}\|\exp[\mathrm{i}S] = \int \mathcal{D}x\mathcal{D}\eta\exp[\mathrm{i}\tilde{S}] \tag{24}$$

where we have introduced anticommuting variables η write the Liouville measure factor $\mathrm{Pf}\ \|\omega_{ab}\|$ as path integral. Now have the following action

$$\tilde{S}_0 = \int_0^T \mathrm{d}t\left[\vartheta_a\dot{x}^a - H + \frac{1}{2}\omega_{ab}\eta^a\eta^b\right]. \tag{25}$$

The systems $(\mathcal{M},H_\alpha,\Omega_\alpha)$ have the classical actions

$$\begin{aligned}
S_0 &= \int_0^T \mathrm{d}t[\vartheta_a\dot{x}^a - H + \frac{1}{2}\theta^a g_{ab}\frac{D\theta^b}{\mathrm{d}t} + \frac{1}{2}\tilde{\omega}_{ab}\theta^a\theta^b] \\
S_2 &= \int_0^T \mathrm{d}t[g_{ab}\chi_H^b\dot{x}^a - \frac{1}{2}g_{ab}\chi_H^a\chi_H^b + \frac{1}{2}\theta^a g_{ab}\frac{D\theta^b}{\mathrm{d}t} + \tilde{\omega}_{ab}\theta^a\theta^b]
\end{aligned} \tag{26}$$

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where

$$\frac{D\theta^b}{dt} = \frac{d\theta^b}{dt} + \dot{x}^d \Gamma^b_{dc} \theta^c \qquad (27)$$

is the covariant derivative along the curve $x(t)$. The corresponding path integral becomes, using the superspace Liouville measure determined by Ω_α

$$Z_\alpha = \int \mathcal{D}x\mathcal{D}\theta \frac{\mathrm{Pf}\,\|\omega_{(\alpha)ab} + R_{abcd}\theta^c\theta^d\|}{\sqrt{\mathrm{Det}\,\|g_{ab}\|}} \exp[\mathrm{i}S_\alpha] = \int \mathcal{D}x\mathcal{D}\theta\mathcal{D}F\mathcal{D}\eta \exp[\mathrm{i}\tilde{S}_\alpha] \qquad (28)$$

where $\tilde{S}_\alpha$ denotes now the quantum actions

$$
\begin{aligned}
\tilde{S}_0 &= \int_0^T dt[\vartheta_a \dot{x}^a - H + \frac{1}{2}\theta^a g_{ab}\frac{D\theta^b}{dt} + \frac{1}{2}\tilde{\omega}_{ab}\theta^a\theta^b + \frac{1}{2}\omega_{ab}\eta^a\eta^b \\
&+ \frac{1}{2}R_{abcd}\eta^a\eta^b\theta^c\theta^d + \frac{1}{2}g_{ab}F^aF^b] \\
\tilde{S}_2 &= \int_0^T dt[g_{ab}\chi_H^b\dot{x}^a - \frac{1}{2}g_{ab}\chi_H^a\chi_H^b + \frac{1}{2}\theta^a g_{ab}\frac{D\theta^b}{dt} + \frac{1}{2}\tilde{\omega}_{ab}\theta^a\theta^b + \frac{1}{2}\tilde{\omega}_{ab}\eta^a\eta^b \\
&+ \frac{1}{2}R_{abcd}\eta^a\eta^b\theta^c\theta^d + \frac{1}{2}g_{ab}F^aF^b]
\end{aligned}
\qquad (29)
$$

Here we have again introduced anticommuting variables η^a and commuting ones F_a to exponentiate the determinants. In particular, if $H = 0$ this almost reduces to the $N = 1$ nonlinear supersymmetric σ-model, up to kinetic term $\frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b which is missing$. We shall discuss the relation of this action to the non-linear σ-model later. In addition we have coupling to symplectic potentials ϑ_a and $g_{ab}\chi_H^b$ which can be interpreted as external gauge fields.

We shall now interpret the path integrals using super loop space equivariant cohomology. In the following we concentrate only on the path integrals for (M, ω, H) and $(\mathcal{M}, \Omega_0, \mathcal{H})$. The reasoning is similar for the action $\tilde{S}_2$ with the replacements $H \to K$ and $\omega \to \tilde{\omega}$. To do this we introduce exterior derivatives in $L\mathcal{M}$. Half of the variables are interpreted as coordinates and the other half as 1-forms. Parameter integrations will not be explicitly written. The first derivative is the equivariant exterior derivative in the loop space relevant to the path integral for (25) whose bosonic part determines a loop space vector field $\chi_S = \dot{x} - \chi_H$ whose zeroes define the Hamilton's equations of motion. We identify the anticommuting variables η as 1-forms and define the basis for contractions

$$
\begin{aligned}
\eta^a &\sim dx^a\,, \\
\iota_a\eta^b &= \delta_a^b\,.
\end{aligned}
\qquad (30)
$$

We define the loop space equivariant exterior derivative

$$d_S = d + \iota_S = \eta^a\frac{\delta}{\delta x^a} + \chi_S^a\iota_a = \eta^a\frac{\delta}{\delta x^a} + (\dot{x}^a - \chi_H^a)\iota_a \qquad (31)$$

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which squares to the loop space Lie derivative $\mathcal{L}_S \sim \mathrm{d}/\mathrm{d}t - \mathcal{L}_H$. This is effectively nilpotent on periodic loops and on the invariant subcomplex of equivariant differential forms. We may use the freedom to make canonical transformations for the symplectic potential $\vartheta \to \vartheta + \mathrm{d}\psi$ such that $\mathcal{L}_H \vartheta = 0$. Using this we can identify $H = \iota_H \vartheta$ which allows us to write the entire action in a form of a cohomological theory

$$\tilde{S} = \mathrm{d}_S \vartheta \ . \tag{32}$$

The other exterior derivative is a proper super loop space exterior derivative (truly nilpotent) obtained by considering x, η as coordinates and θ, F their differentials

$$\delta = \theta^a \frac{\delta}{\delta x^a} + F^a \frac{\delta}{\delta \eta^a} \ . \tag{33}$$

Now the actions (29) are obtained by $\tilde{S}_0 = \tilde{S} + \delta\Theta$ with

$$\Theta = \frac{1}{2} g_{ab} (\dot{x}^a - \chi^a) \theta^a + \frac{1}{2} \Gamma^a_{bc} \eta^b \eta^c \theta_a + \frac{1}{2} g_{ab} \tilde{F}^a \eta^b \tag{34}$$

where $\tilde{F}^a = F^a - \Gamma^a_{bc} \eta^b \theta^c$. On the classical level this is consistent with the Poisson bracket relations on M and $\mathcal{M}$ since the addition of an (locally) exact piece should not change the equations of motion obtained from the action principle. However, the path integrals will be different since we have a different number of degrees of freedom.

Now we shall first evaluate the path integrals exactly using localization techniques [6] based on equivariant cohomology in super loop space. The evaluation relies on the Lie-derivative condition for the metric. We first concentrate on the path integral (25). The action $\tilde{S}$ is d_S-closed and therefore the path integral remains intact under $\tilde{S} \to \tilde{S} + \lambda \mathrm{d}_S \Psi$ whenever the 1-form (gauge fermion) Ψ satisfies $\mathrm{d}_S^2 \Psi = \mathcal{L}_S \Psi = 0$. The limit $\lambda \to 0$ gives the original path integral and $\lambda \to \infty$ gives a localization to certain configurations, depending on Ψ.

To construct a gauge fermion Ψ we also need a metric in M. Under the assumption that χ_H is a Killing vector for g_{ab} the following gauge fermions satisfy (11), as we have assumed. $\Psi = \frac{1}{2} g_{ab} \dot{x}^a \eta^b$ reduces the path integral to an ordinary integral over M, producing the result

$$Z = \int \mathrm{d}x \mathrm{d}\eta \exp[-\mathrm{i}T(H - \omega)] \sqrt{\mathrm{Det}\left[\frac{\mathcal{R}/2}{\sin(T\mathcal{R}/2)}\right]} = \int_M \mathrm{Ch}[-\mathrm{i}T(H - \omega)] \hat{A}[T\mathcal{R}] \tag{35}$$

where $\mathcal{R} = \|R^a_b + \tilde{\Omega}^a_b\|$ is the equivariant curvature of M. The symbols Ch and $\hat{A}$ denote the (equivariant) Chern class and the Dirac genus. To obtain this result we have separated x, η to constant and non-constant modes and scaled the latter ones

$$x \ = \ x_0 + \frac{1}{\sqrt{\lambda}} x_t$$

$$\eta \;=\; \eta_0 + \frac{1}{\sqrt{\lambda}}\eta_t \tag{36}$$

and integrated over non-constant modes. The Jacobian for this transformation is trivial. The expression (35) is an equivariant generalization for the Atiyah-Singer index for a Dirac operator on a Riemannian manifold, reducing to the standard index when $H = 0$.

We shall now consider the path integral for the system $(\mathcal{M}, \Omega_\alpha, H_\alpha)$ [8]. We interpret x^a and θ_a as coordinates in the super loop space and η^a and F_a as their differentials and introduce corresponding contractions

$$
\begin{aligned}
\eta^a &\sim & \mathrm{d}x^a\,, \\
F_a &\sim & \mathrm{d}\theta_a\,, \\
\iota_a \eta^b &=& \delta_a^b \\
\pi^a F_b &=& \delta_b^a\,.
\end{aligned}
\tag{37}
$$

We define the super loop space equivariant exterior derivative

$$Q = \eta^a \frac{\delta}{\delta x^a} + F_a a \frac{\delta}{\delta \theta_a} + (\dot{x}^a - \chi_H^a)\iota_a + (\dot{\theta}_b + \partial_b \chi_H^a \theta_a)\pi^b \tag{38}$$

with $Q^2 = \mathcal{L}_S = \mathrm{d}/\mathrm{d}t - \mathcal{L}_H$. We introduce a canonical conjugation $Q \to e^{-\Phi}Q_S e^{\Phi} = \tilde{Q}$ which does not change its cohomology. Choosing the functional $\Phi = \Gamma_{bc}^a \pi^b \eta^c \theta_a$ we get

$$
\begin{aligned}
\tilde{Q} &=& \eta^a \frac{\delta}{\delta x^a} + F_a \frac{\delta}{\delta \theta_a} + (\dot{x}^a - \chi_H^a)\iota_a + (\dot{\theta}_a + \partial_a \chi_H^b \theta_b)\pi^a \\
&+& \left(\Gamma_{bc}^a F_a \eta^b - \frac{1}{2} R_{bcd}^a \eta^c \eta^d \theta_a - (\dot{x}^b - \chi_H^b)\Gamma_{bc}^a \theta_a + (\delta_b^a \partial_t + \partial_a \chi_H^b)\theta_c \right) \pi^c\,.
\end{aligned}
\tag{39}
$$

The pertinent action (29) can be obtained from the 2-dimensional $N = 1$ supersymmetric σ-model by partial localization to a 1-dimensional model [8]. In this procedure the kinetic term for the bosons (in light-cone coordinates) $g_{ab}\partial_+ \phi^a \partial_- \phi^b$ drops out. The action $\tilde{S}_0$ is $\tilde{Q}$-closed, $\tilde{Q}\tilde{S}_0 = 0$, and therefore the path integral Z_0 remains intact if we add a term $\tilde{Q}\Psi$ to the action. We shall consider the following gauge fermions. $\Psi_1 = g_{ab}F^a\theta^b + \frac{\lambda}{2}g_{ab}(\dot{x}^a - \chi_H^a)\eta^b$ localizes the path integral in the limit $\lambda \to \infty$ to the T-periodic classical trajectories for the original action (23)

$$Z_0 = \sum_{\delta S = 0} \mathrm{sign}\left[\mathrm{Det}\,\|\frac{\delta^2 S}{\delta x^a \delta x^b}\|\right]\exp[\mathrm{i}S] \tag{40}$$

which can be interpreted as a loop space generalization of the Poincáre-Hopf theorem. By selecting $\Psi_2 = g_{ab}F^a\theta^b + \frac{\lambda}{2}g_{ab}\dot{x}^a\eta^b$ we find that the path integral localizes to an ordinary integral over M

$$Z_0 = \int \mathrm{d}x\,\mathrm{d}\eta \exp[-\mathrm{i}T(H - \frac{1}{2}\omega_{ab}\eta^a\eta^b)]\mathrm{Pf}\left[\frac{1}{2}(\tilde{\omega}_b^a + R_{bcd}^a \eta^c \eta^d)\right]$$

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$$= \int_M \mathrm{Ch}[-iT(H-\omega)]\mathrm{Eul}\,[\mathcal{R}] \qquad (41)$$

which is an equivariant generalization of the Gauss-Bonnet theorem for the Euler characteristic of M. Indeed, setting, $H = \vartheta = 0$ we obtain the standard formula

$$\chi(M) = \int_M \mathrm{Eul}(R) = \sum_{\mathrm{d}H=0} \mathrm{sign}\,[\det \|\partial_a \partial_b H\|]\,. \qquad (42)$$

We now discuss some aspects of out results. The localization of the path integral (22) is related to the loop space generalization of the Duistermaat-Heckman theorem, to equivariant index and Lefschetz fixed point theorems. For example, applications to path integral proofs for index theorems have been considered in [9]. In particular, it can be used to evaluate exactly the path integral related to the quantization of coadjoint orbits of semisimple Lie groups. In [10] it was shown to exactly yield the correct character for $SU(2)$ without the Weyl shift by replacing the $\hat{A}$-genus by Todd-genus.

In summary, we have studied Hamiltonian systems on ordinary symplectic manifolds and supermanifolds. On supermanifolds we found both odd and even symplectic structures for Hamiltonians generating isometries. The existence of bi-Hamiltonian structures implied the classical integrability of the systems. The models on the supermanifold also turned out to be closely related to the supersymmetric non-linear σ-model. It was possible to quantize the systems using the even symplectic structure and evaluate the path integrals exactly. The results were equivariant generalizations of familiar topological invariants.

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